Lecture 25

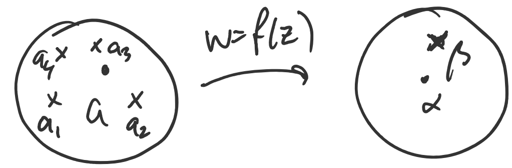
Friday, December 6, 2019 6:00 AM

Recall. • Argument principle. Assume $\gamma \approx 0$ and f anal. in G. If $f(z)=\alpha$ has roots $a_1, \dots, a_n$ in G, then $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)-\alpha} = \sum_{k=1}^{n} n(\gamma, a_k)$.

Typical application: $\gamma \approx 0$ is simple w/ $\mathbb{C}\setminus\{\gamma\} = \{n(\gamma,z)=0\} \cup \underbrace{\{n(\gamma,z)=1\}}_{G_1 \subseteq G \text{ since } \gamma \approx 0}$

Then, $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)-\alpha} = \#\{\text{roots w/ multi in } G_1\}$

• Local behavior of $w=f(z)$. Suppose $f(z)=\alpha$ has root of multi $m \geq 1$ at $a \in G$. Then, $\exists\, \varepsilon, \delta > 0$ s.t. $f(z)=\beta$ has m simple (multi $=1$) roots in $B(a,\delta)$ for each $\beta \in B(\alpha, \varepsilon)$.

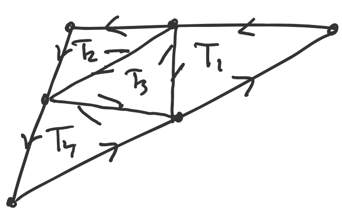


In particular, $B(\alpha,\varepsilon) \subseteq f(B(a,\delta))$. Thus, if G is a region, f nonconstant
$\Rightarrow$ for every $\alpha \in f(G)$ and $a \in f^{-1}(\alpha)$, $f(z)=\alpha$ has root of finite multi at $z=a$.
$\Rightarrow \exists\, B(\alpha,\varepsilon) \subseteq f(G) \Rightarrow f(G)$ open.

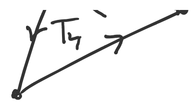
• Do Open Mapping Thm + Cor 1 from Lecture 24 notes.

Goursat's Thm. Let $G \subseteq \mathbb{C}$ and assume f is $\mathbb{C}$-diff. at every $a \in G$. Then f is anal. in G.

Pf. Use Morera's Thm. Pick $B(a,r) \subseteq G$, and triangular path $T \subseteq B(a,r)$.



$T = \partial\Delta$, Δ closed triangle.



We have $\displaystyle\int_T f\,dz = \sum_{k=1}^{4} \int_{T_k} f\,dz$ by cancellation over interior segments. Pick $\underset{=\partial\Delta^{(1)}}{T^{(1)}}$ to be T_k s.t. $\displaystyle\left|\int_{T^{(1)}} f\,dz\right| = \max_k \left|\int_{T_k} f\,dz\right|$

$$\Rightarrow \left|\int_T f\right| \leq 4\left|\int_{T^{(1)}} f\right|.$$ <u>Note</u>: $\ell(T^{(1)}) = \frac{1}{2}\ell(T)$, $\operatorname{diam}(\Delta^{(1)}) = \frac{1}{2}\operatorname{diam}(\Delta)$.

Repeat. Inductively, we obtain $\Delta^{(0)} = \Delta \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \ldots$ closed triangles. We have $\ell(T^{(n)}) = 2^{-n}\ell(T)$, $\operatorname{diam}\Delta^{(n)} = 2^{-n}\operatorname{diam}\Delta$. By Cantor's Thm $\displaystyle\bigcap_{n=0}^{\infty} \Delta^{(n)} = \{z_0\}$.

Since f has $\mathbb{C}$-der. $f'(z_0)$, $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0| \quad , \ |z - z_0| < \delta .$$

Now, $h(z) = f(z_0) + f'(z_0)(z - z_0)$ is anal. (linear in z) $\Rightarrow$

$$\int_{T^{(n)}} f(z)\,dz = \int_{T^{(n)}} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\,dz \Rightarrow$$

$$\left|\int_T f\,dz\right| \leq 4^n \left|\int_{T^{(n)}} f\,dz\right| = 4^n \left|\int_{T^{(n)}} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\,dz\right| \leq \left\{\begin{matrix}\text{choose } n \text{ s.t.} \\ \operatorname{diam}\Delta^{(n)} < \delta\end{matrix}\right\}$$

$$\leq 4^n \cdot \varepsilon \underbrace{|z - z_0|}_{\leq \operatorname{diam}\Delta^{(n)}} \cdot \ell(T^{(n)}) \leq \varepsilon \cdot 4^n \cdot 2^{-n}\operatorname{diam}(\Delta) \cdot 2^{-n}\ell(T)$$

$$= \varepsilon \operatorname{diam}(\Delta)\,\ell(T).$$

Since ε arbitrary $\Rightarrow \displaystyle\int_T f\,dz = 0 \Rightarrow f$ anal. by Morera. ☒